

The Perturbation Technique for finding “Marginal” Equations, and other useful things.

1 Perturbation Technique Applied to non-linear Demand Curves

Consider a demand curve of this shape:

$$Q = a/P$$

Suppose that we’re interested in finding the elasticity of this demand curve. Remember, elasticity is of the form:

$$E_d = \frac{P}{Q} \frac{\Delta Q}{\Delta P}$$

When we face a linear demand function, it’s easy to find $\frac{\Delta Q}{\Delta P}$. However, this is most definitely NOT a linear demand function. So, how can we find it in this case? Here’s a technique that will work.

1. Replace each variable X with “X + Δ X” (Note: treat “Δ X” as a DIFFERENT variable than X.)
2. Solve the equation for the ratio you want.
3. If there are stray Δ X when you have the ratio by itself on one side of the equation, act like those are zero. (Since we’re usually looking for these things at a point, it makes sense to assume the change to be very small, and there’s not much difference between “very small” and “zero”.)

So, let’s apply the technique here.

Step 1:

$$Q + \Delta Q = a / (P + \Delta P)$$

Step 2:

Lots of algebra here. First, I’m going to get rid of “Q” by substituting in the original equation.

$$a/P + \Delta Q = a / (P + \Delta P)$$

Now, let’s get rid of fractions. I hate fractions. So, I need to multiply both sides by P first.

$$a + P\Delta Q = (aP) / (P + \Delta P)$$

Now, by (P + Δ P).

$$(a + P\Delta Q)(P + \Delta P) = aP$$

Now, let’s distribute, because I hate quantities.

$$aP + P^2\Delta Q + a\Delta P + P\Delta Q\Delta P = aP$$

Ah, we can cancel the aP, as it appears on both sides.

$$P^2\Delta Q + a\Delta P + P\Delta Q\Delta P = 0$$

Now, let’s get all of the ΔQ on one side, by itself.

$$P^2 \Delta Q + P \Delta Q \Delta P = -a \Delta P$$

Now, to get the ratio we want, divide by ΔP

$$P^2 \frac{\Delta Q}{\Delta P} + P \Delta Q = -a$$

To make sure that the ratio is totally alone, let's move the other on the left over to the right, and divide by P^2

$$\frac{\Delta Q}{\Delta P} = -a/P^2 - \Delta Q/P$$

Alright, now we have the ratio all solved. Just one thing left to do... act like ΔQ on the right hand side is zero. Then, we have:

$$\frac{\Delta Q}{\Delta P} = -a/P^2$$

Tah dah!

Now, we can plug that into the elasticity equation...

$$E_d = \frac{P}{Q} \frac{a}{P^2}$$

Now, with a quick substitution for the demand curve...

$$E_d = \frac{P}{a/P} \frac{a}{P^2}$$

Funny... Let me multiply top and bottom of the first fraction by P...

$$E_d = \frac{P^2}{a} \frac{a}{P^2}$$

Interesting, it looks like, once we simplify things...

$$E_d = 1$$

This form gives us a demand curve that is unit elastic, regardless where we are on it! (Also, what "a" is doesn't seem to matter for elasticity!)

Note: this provides the same answer as if we were using calculus to find the same thing, we just get there with algebra.

2 Perturbation applied to Marginal Revenue

We can use the same technique when calculating marginal revenue for a monopolist.

Say that demand is $Q = a - bP$.

$$TR = PQ.$$

$$MR = \frac{\Delta TR}{\Delta Q}$$

Since marginal revenue requires a change in quantity, it makes sense to rearrange the demand curve so that we can substitute price out of the total revenue equation. So...

$$P = (a - Q)/b$$

Now, we can substitute that into the total revenue equation.

$$TR = (a-Q)/b * Q$$

Now, using our technique...

Step 1:

$$TR + \Delta TR = (a-(Q+\Delta Q))/b * (Q + \Delta Q)$$

Step 2:

Let's get rid of "TR" by substituting...

$$(a-Q)/b * Q + \Delta TR = (a-(Q+\Delta Q))/b * (Q + \Delta Q)$$

Now, let's get rid of fractions by multiplying by b.

$$(a-Q) * Q + b \Delta TR = (a-(Q+\Delta Q)) * (Q + \Delta Q)$$

Now, let's distribute everything.

$$aQ - Q^2 + b \Delta TR = aQ + a \Delta Q - Q^2 - Q \Delta Q - Q \Delta Q - (\Delta Q)^2$$

Ooo, we can eliminate a couple terms and combine a couple others.

$$b \Delta TR = a \Delta Q - 2Q \Delta Q - (\Delta Q)^2$$

Looks like we have to divide by b to get ΔTR by itself.

$$\Delta TR = a \Delta Q/b - 2Q \Delta Q/b - (\Delta Q)^2/b$$

Now, divide by ΔQ .

$$\frac{\Delta TR}{\Delta Q} = a/b - 2Q/b - \Delta Q/b$$

Got it!

Step 3: Just have that pesky ΔQ to be rid of.

$$\frac{\Delta TR}{\Delta Q} = a/b - 2Q/b$$

And that's it!

Look at that equation. Remember it. Why? Because it's a *general form that will usually work*. All you need is a single condition:

$$\text{Demand is of the form: } Q = a - bP$$

(Note: if demand is of the form: $P = a - bQ$, then this formula will NOT work. However, you can use the perturbation technique to solve for a formula that will work in that case, if you like. The formula you're looking for is: $\frac{\Delta TR}{\Delta Q} = a - 2bQ$. See if you can derive it.)

3 Perturbation and Cost

Now, let's try it with marginal cost!

Say we have a cost function that looks like this:

$$TC = aQ^3 + FC$$

(FC is a fixed cost - that is it does not change, and should NOT be considered a variable.)

Let's use our technique to find $MC = \frac{\Delta TC}{\Delta Q}$

Step 1:

$$TC + \Delta TC = a(Q + \Delta Q)^3 + FC$$

Remember, FC is NOT a variable, so it doesn't get a Δ term!

Step 2:

Let's do two things: first, I'll substitute for TC. Also, I'll write out the cubic term more extensively.

$$aQ^3 + FC + \Delta TC = a(Q + \Delta Q)(Q + \Delta Q)(Q + \Delta Q) + FC$$

Ah, we can already get rid of FC - which proves that fixed costs have NO impact on marginal cost.

(We could do this in a more general form to prove it more generally, but I'll leave that to you.) Also, let's distribute the first two terms in that nasty thing on the right hand side.

$$aQ^3 + \Delta TC = a(Q^2 + Q \Delta Q + Q \Delta Q + (\Delta Q)^2)(Q + \Delta Q)$$

Now, combine terms in that big parenthetical expression.

$$aQ^3 + \Delta TC = a(Q^2 + 2Q \Delta Q + (\Delta Q)^2)(Q + \Delta Q)$$

Now, let's distribute again.

$$aQ^3 + \Delta TC = a(Q^3 + Q^2 \Delta Q + 2Q^2 \Delta Q + 2Q (\Delta Q)^2 + Q (\Delta Q)^2 + (\Delta Q)^3)$$

Now, put that a through the whole thing and combine terms.

$$aQ^3 + \Delta TC = aQ^3 + 3aQ^2 \Delta Q + 3aQ (\Delta Q)^2 + a(\Delta Q)^3$$

We get to eliminate another term!

$$\Delta TC = 3aQ^2 \Delta Q + 3aQ (\Delta Q)^2 + a(\Delta Q)^3$$

Step 3:

$$\frac{\Delta TC}{\Delta Q} = 3aQ^2 + 3aQ (\Delta Q) + a(\Delta Q)^2$$

And then, treat (ΔQ) like zero...

$$\frac{\Delta TC}{\Delta Q} = 3aQ^2$$

And we have it!